When G is triangle-free, the faces have length at least 4. In this case $2e = \sum f_i \geq 4f$, and we obtain $e \leq 2n - 4$. ■

6.1.24. Example. Nonplanarity of K_5 and $K_{3,3}$ follows immediately from Theorem 6.1.23. For K_5, we have $e = 10 > 9 = 3n - 6$. Since $K_{3,3}$ is triangle-free, we have $e = 9 > 8 = 2n - 4$. These graphs have too many edges to be planar. ■

6.1.25. Definition. A **maximal planar graph** is a simple planar graph that is not a spanning subgraph of another planar graph. A **triangulation** is a simple plane graph where every face boundary is a 3-cycle.

6.1.26. Proposition. For a simple n-vertex plane graph G, the following are equivalent.
A) G has $3n - 6$ edges.
B) G is a triangulation.
C) G is a maximal plane graph.

Proof: A $\Leftrightarrow$ B. For a simple n-vertex plane graph, the proof of Theorem 6.1.23 shows that having $3n - 6$ edges is equivalent to $2e = 3f$, which occurs if and only if every face is a 3-cycle.

B $\Leftrightarrow$ C. There is a face that is longer than a 3-cycle if and only if there is a way to add an edge to the drawing and obtain a larger simple plane graph. ■

6.1.27. Remark. A graph embeds in the plane if and only if it embeds on a sphere. Given an embedding on a sphere, we can puncture the sphere inside a face and project the embedding onto a plane tangent to the opposite point. This yields a planar embedding in which the punctured face on the sphere becomes the unbounded face in the plane. The process is reversible. ■

6.1.28. Application. *Regular polyhedra.* Informally, we think of a regular polyhedron as a solid whose boundary consists of regular polygons of the same length, with the same number of faces meeting at each vertex. When we expand the polyhedron out to a sphere and then lay out the drawing in the plane as in Remark 6.1.27, we obtain a regular plane graph with faces of the same length. Hence the dual also is a regular graph.

Let G be a plane graph with n vertices, e edges, and f faces. Suppose that G is regular of degree k and that all faces have length l. The degree-sum formula for G and for G^* yields $kn = 2e = lf$. By substituting for n and f in Euler's Formula, we obtain $e(\frac{2}{k} - 1 + \frac{2}{l}) = 2$. Since e and 2 are positive, the other factor must also be positive, which yields $(2/k) + (2/l) > 1$, and hence $2l + 2k > kl$. This inequality is equivalent to $(k-2)(l-2) < 4$.

Because the dual of a 2-regular graph is not simple, we require that $k, l \geq 3$. Now $(k-2)(l-2) < 4$ also requires $k, l \leq 5$. The only integer pairs satisfying these requirements for (k, l) are $(3,3)$, $(3,4)$, $(3,5)$, $(4,3)$, and $(5,3)$.

Once we specify k and l, there is only one way to lay out the plane graph when we start with any face. Hence there are only the five Platonic solids listed below, one for each pair (k, l) that satisfying the requirements. ■

k	l	$(k-2)(l-2)$	e	n	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	icosahedron

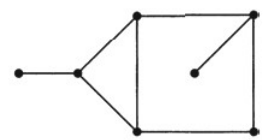
EXERCISES

6.1.1. (−) Prove or disprove:
a) Every subgraph of a planar graph is planar.
b) Every subgraph of a nonplanar graph is nonplanar.

6.1.2. (−) Show that the graphs formed by deleting one edge from K_5 and $K_{3,3}$ are planar.

6.1.3. (−) Determine all r, s such that $K_{r,s}$ is planar.

6.1.4. (−) Determine the number of isomorphism classes of planar graphs that can be obtained as planar duals of the graph below



6.1.5. (−) Prove that a plane graph has a cut-vertex if and only if its dual has a cut-vertex.

6.1.6. (−) Prove that a plane graph is 2-connected if and only if for every face, the bounding walk is a cycle.

6.1.7. (−) A **maximal outerplanar graph** is a simple outerplanar graph that is not a spanning subgraph of a larger simple outerplanar graph. Let G be a maximal outerplanar graph with at least three vertices. Prove that G is 2-connected.

6.1.8. (−) Prove that every simple planar graph has a vertex of degree at most 5.

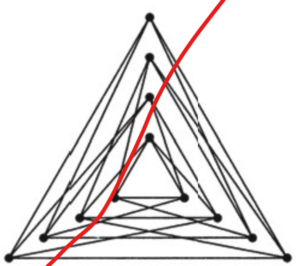
6.1.9. (−) Use Theorem 6.1.23 to prove that every simple planar graph with fewer than 12 vertices has a vertex of degree at ~~most~~ 4.

6.1.10. (−) Prove or disprove: There is no simple bipartite planar graph with minimum degree at least 4.

6.1.11. (−) Let G be a maximal planar graph. Prove that G^* is 2-edge-connected and 3-regular.

6.1.12. (−) Draw the five regular polyhedra as planar graphs. Show that the octahedron is the dual of the cube and the icosahedron is the dual of the dodecahedron.

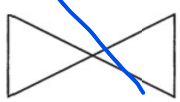
6.1.13. Find a planar embedding of the graph below.



6.1.14. Prove or disprove: For each $n \in \mathbb{N}$, there is a simple connected 4-regular planar graph with more than n vertices.

6.1.15. Construct a 3-regular planar graph of diameter 3 with 12 vertices. (Comment: T. Barcume proved that no such graph has more than 12 vertices.)

6.1.16. Let F be a figure drawn continuously in the plane without retracing any segment, ending at the start (this can be viewed as an Eulerian graph). Prove that F can be drawn without allowing the pencil point to cross what has already been drawn. For example, the figure below has two traversals; one crosses itself and the other does not.



6.1.17. Prove or disprove: If G is a 2-connected simple plane graph with minimum degree 3, then the dual graph G^* is simple.

6.1.18. Given a plane graph G, draw the dual graph G^* so that each dual edge intersects its corresponding edge in G and no other edge. Prove the following.
a) G^* is connected.
b) If G is connected, then each face of G^* contains exactly one vertex of G.
c) $(G^*)^* = G$ if and only if G is connected.

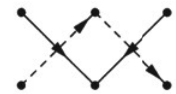
6.1.19. Let G be a plane graph. Use induction on $e(G)$ to prove Theorem 6.1.14: a set $D \subseteq E(G)$ is a cycle in G if and only if the corresponding set $D^* \subseteq E(G^*)$ is a bond in G^*. (Hint: Contract an edge of D and apply Remark 6.1.15.)

6.1.20. Prove by induction on the number of faces that a plane graph G is bipartite if and only if every face has even length.

6.1.21. (!) Prove that a set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*.

6.1.22. The **weak dual** of a plane graph G is the graph obtained from the dual G^* by deleting the vertex for the unbounded face of G. Prove that the weak dual of an outerplane graph is a forest.

6.1.23. (!) *Directed plane graphs.* Let G be a plane graph, and let D be an orientation of G. The **dual** D^* is an orientation of G^* such that when an edge of D is traversed from tail to head, the dual edge in D^* crosses it from right to left. For example, if the solid edges below are in D, then the dashed edges are in D^*.



Prove that if D is strongly connected, then D^* has no cycle, and $\delta^-(D^*) = \delta^+(D^*) = 0$. Conclude that if D is strongly connected, then D has a face on which the edges form a clockwise cycle and another face on which the edges form a counterclockwise cycle.

6.1.24. (!) *Alternative proof of Euler's Formula.*
a) Use polygonal curves (not Euler's Formula) to prove by induction on $n(G)$ that every planar embedding of a tree G has one face.
b) Prove Euler's Formula by induction on the number of cycles.

6.1.25. (!) Prove that every n-vertex plane graph isomorphic to its dual has $2n-2$ edges. For all $n \ge 4$, construct a simple n-vertex plane graph isomorphic to its dual.

6.1.26. Determine the maximum number of edges in a simple outerplane graph with n vertices, giving three proofs.
a) By induction on n.
b) By using Euler's Formula.
c) By adding a vertex in the unbounded face and using Theorem 6.1.23.

6.1.27. Let G be a connected 3-regular plane graph in which every vertex lies on one face of length 4, one face of length 6, and one face of length 8.
a) In terms of $n(G)$, determine the number of faces of each length.
b) Use Euler's Formula and part (a) to determine the number of faces of G.

6.1.28. Let C be a closed curve bounding a convex region in the plane. Suppose that m chords of C are drawn so that no three share a point and no two share an endpoint. Let p be the number of pairs of chords that cross. In terms of m and p, compute the number of segments and the number of regions formed inside C. (Alexanderson–Wetzel [1977])

6.1.29. Prove that the complement of a simple planar graph with at least 11 vertices is nonplanar. Construct a self-complementary simple planar graph with 8 vertices.

6.1.30. (!) Let G be an n-vertex simple planar graph with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges. Use this to prove that the Petersen graph is nonplanar.

6.1.31. Let G be the simple graph with vertex set $v_1, \dots, v_n$ whose edges are $\{v_i v_j \colon |i-j| \le 3\}$. Prove that G is a maximal planar graph.

6.1.32. Let G be a maximal planar graph. Prove that if S is a separating 3-set of G^*, then $G^* - S$ has two components. (Chappell)

6.1.33. (!) Let G be a triangulation, and let n_i be the number of vertices of degree i in G. Prove that $\sum (6-i)n_i = 12$.

6.1.34. Construct an infinite family of simple planar graphs with minimum degree 5 such that each has exactly 12 vertices of degree 5. (Hint: Modify the dodecahedron.)

6.1.35. (!) Prove that every simple planar graph with at least four vertices has at least four vertices with degree less than 6. For each even value of n with $n \ge 8$, construct an n-vertex simple planar graph G that has exactly four vertices with degree less than 6. (Grünbaum–Motzkin [1963])

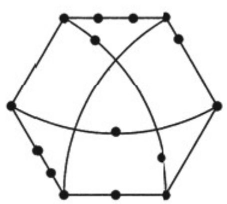
6.1.36. Let S be a set of n points in the plane such that for all $x, y \in S$, the distance in the plane between x and y is at least 1. Prove that there are at most $3n - 6$ pairs u, v in S such that the distance in the plane between u and v is exactly 1.

6.1.37. Given integers $k \geq 2$, $l \geq 1$, and kl even, construct a planar graph with exactly k faces in which every face has length l.

6.2. Characterization of Planar Graphs

Which graphs embed in the plane? We have proved that K_5 and $K_{3,3}$ do not. In fact, these are the crucial graphs and lead to a characterization of planar graphs known as Kuratowski's Theorem. Kasimir Kuratowski once asked Frank Harary about the origin of the notation for K_5 and $K_{3,3}$. Harary replied, "The K in K_5 stands for Kasimir, and the K in $K_{3,3}$ stands for Kuratowski!"

Recall that a subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths (Definition 5.2.19).



a subdivision of $K_{3,3}$

6.2.1. Proposition. If a graph G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is nonplanar.

Proof: Every subgraph of a planar graph is planar, so it suffices to show that subdivisions of K_5 and $K_{3,3}$ are nonplanar. Subdividing edges does not affect planarity; the curves in an embedding of a subdivision of G can be used to obtain an embedding of G, and vice versa. ■

By Proposition 6.2.1, avoiding subdivisions of K_5 and $K_{3,3}$ is a necessary condition for being a planar graph. Kuratowski proved TONCAS:

6.2.2. Theorem. (Kuratowski [1930]) A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. ■

Kuratowski's Theorem is our goal in the first half of this section, after which we will comment on other characterizations of planar graphs.

When G is planar, we can seek a planar embedding with additional properties. Wagner [1936], Fáry [1948], and Stein [1951] showed that every finite

simple planar graph has an embedding in which all edges are straight line segments; this is known as **Fáry's Theorem** (Exercise 6). For 3-connected planar graphs, we will prove the stronger property that there exists an embedding in which every face is a convex polygon.

PREPARATION FOR KURATOWSKI'S THEOREM

We introduce short names for subgraphs that demonstrate nonplanarity.

6.2.3. Definition. A **Kuratowski subgraph** of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A **minimal nonplanar graph** is a nonplanar graph such that every proper subgraph is planar.

We will prove that a minimal nonplanar graph with no Kuratowski subgraph must be 3-connected. Showing that every 3-connected graph with no Kuratowski subgraph is planar then completes the proof of Kuratowski's Theorem.

6.2.4. Lemma. If F is the edge set of a face in a planar embedding of G, then G has an embedding with F being the edge set of the unbounded face.

Proof: Project the embedding onto the sphere, where the edge sets of regions remain the same and all regions are bounded, and then return to the plane by projecting from inside the face bounded by F. ■

6.2.5. Lemma. Every minimal nonplanar graph is 2-connected.

Proof: Let G be a minimal nonplanar graph. If G is disconnected, then we embed one component of G inside one face of an embedding of the rest.

If G has a cut-vertex v, let $G_1, \dots, G_k$ be the $\{v\}$-lobes of G. By the minimality of G, each G_i is planar. By Lemma 6.2.4, we can embed each G_i with v on the outside face. We squeeze each embedding to fit in an angle smaller than $360/k$ degrees at v, after which we combine the embeddings at v to obtain an embedding of G. ■

6.2.6. Lemma. Let $S = \{x, y\}$ be a separating 2-set of G. If G is nonplanar, then adding the edge xy to some S-lobe of G yields a nonplanar graph.

Proof: Let $G_1, \dots, G_k$ be the S-lobes of G, and let $H_i = G_i \cup xy$. If H_i is planar, then by Lemma 6.2.4 it has an embedding with xy on the outside face. For each $i > 1$, this allows H_i to be attached to an embedding of $\bigcup_{j=1}^{i-1} H_j$ by embedding H_i in a face that has xy on its boundary. Afterwards, deleting the edge xy if it is not in G yields a planar embedding of G. ■

The next lemma allows us to restrict our attention to 3-connected graphs in order to prove Kuratowski's Theorem. The hypothesized graph doesn't exist, but if it did, it would be 3-connected.

6.2.19. Theorem. (Demoucron–Malgrange–Pertuiset [1964]) Algorithm 6.2.17 produces a planar embedding if G is planar.

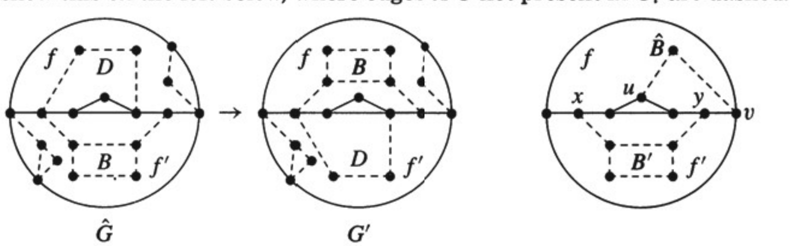
Proof: We may assume that G is 2-connected. A cycle appears as a simple closed curve in every planar embedding. Since we can reflect the plane, every embedding of a cycle in a planar graph G extends to an embedding of G.

Hence G_0 extends to a planar embedding of G if G is planar. It suffices to show that if the plane graph G_i is extendable to a planar embedding of G and the algorithm produces a plane graph G_{i+1} from G_i, then G_{i+1} also is extendable to a planar embedding of G. Note that every G_i-fragment has at least two vertices of attachment, since G is 2-connected.

If some G_i-fragment B has $|F(B)| = 1$, then there is only one face of G_i that can contain P in an extension of G_i to a planar embedding of G. The algorithm puts P in that face to obtain G_{i+1}, so in this case G_{i+1} is extendable.

Problems can arise only if $|F(B)| > 1$ for all B and we select the wrong face in which to embed a path P from the selected fragment. Suppose that (1) we embed P in face $f \in F(B)$, and (2) G_i can be extended to a planar embedding $\hat{G}$ of G in which P is inside face $f' \in F(B)$. We modify $\hat{G}$ to show that G_i can be extended to another embedding G' of G in which P is inside f. This shows that our choice causes no problem, and the constructed G_{i+1} is extendable.

Let C be the set of vertices in the boundaries of both f and f'; this includes the vertices of attachment of B. We draw G' by switching between f and f' all G_i-fragments that $\hat{G}$ places in f or f' and whose vertices of attachment lie in C. We show this on the left below, where edges of G not present in G_i are dashed.



The change switches B and produces the desired embedding G' unless some unswitched G_i-fragment $\hat{B}$ conflicts with a switched fragment. Since the switch is symmetric in f and f' and changes only their interiors, we may assume that $\hat{B}$ appears in f in $\hat{G}$. "Conflict" means that $\hat{G}$ has some B' in f', which we are trying to move to f, such that $\hat{B}$ and B' are adjacent in the conflict graph of f.

Let $\hat{A}, A'$ denote the vertex sets where $\hat{B}, B'$ attach to the boundary of f. Since $\hat{B}$ and B' conflict, $\hat{A}, A'$ have three common vertices or four alternating vertices on the boundary of f. Since $A' \subseteq C$ but $\hat{A} \not\subseteq C$, the first possibility implies the second. Let x, u, y, v be the alternation, with $x, y \in A' \subseteq C$ and $u, v \in \hat{A}$. We may assume that $u \notin C$, as shown on the right above; if there is no such alternation, then $\hat{B}, B'$ do not conflict or $\hat{B}$ can switch to f'.

Since $u \notin C$ and y is between u and v on f, no other face contains both u and v. Thus $\hat{B}$ fails to have its vertices of attachment contained in at least two faces, contradicting the hypothesis that $\left|F(\hat{B})\right| > 1$. ■

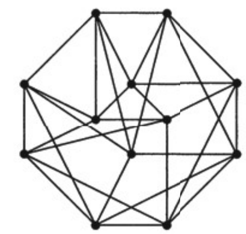
We can begin by checking that G has at most $3n - 6$ edges, maintain appropriate lists for the face boundaries, and perform the other operations via searches of linear size. Thus this algorithm runs in quadratic time. The proof of Kuratowski's Theorem by Klotz [1989] also gives a quadratic algorithm to test planarity, and it finds a Kuratowski subgraph when G is not planar.

EXERCISES

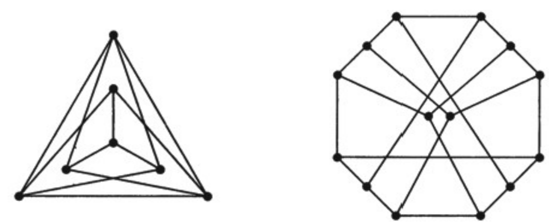
6.2.1. (−) Prove that the complement of the 3-dimensional cube Q_3 is nonplanar.

6.2.2. (−) Give three proofs that the Petersen graph is nonplanar.
a) Using Kuratowski's Theorem.
b) Using Euler's Formula and the fact that the Petersen graph has girth 5.
c) Using the planarity-testing algorithm of Demoucron–Malgrange–Pertuiset.

6.2.3. (−) Find a convex embedding in the plane for the graph below.



6.2.4. (−) For each graph below, prove nonplanarity or provide a convex embedding.



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6.2.5. Determine the minimum number of edges that must be deleted from the Petersen graph to obtain a planar subgraph.

6.2.6. (!) *Fáry's Theorem.* Let R be a region in the plane bounded by a simple polygon with at most five sides (**simple polygon** means the edges are line segments that do not cross). Prove there is a point x inside R that "sees" all of R, meaning that the segment from x to any point of R does not cross the boundary of R. Use this to prove inductively that every simple planar graph has a straight-line embedding.

6.2.7. (!) Use Kuratowski's Theorem to prove that G is outerplanar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$. (Hint: To apply Kuratowski's Theorem, find an appropriate modification of G. This is much easier than trying to mimic a proof of Kuratowski's Theorem.)

6.2.8. (!) Prove that every 3-connected graph with at least six vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$. (Wagner [1937])

6.2.9. (+) For $n \geq 5$, prove that the maximum number of edges in a simple planar n-vertex graph not having two disjoint cycles is $2n-1$. (Comment: Compare with Exercise 5.2.28.) (Markus [1999])

6.2.10. (!) Let $f(n)$ be the maximum number of edges in a simple n-vertex graph containing no $K_{3,3}$-subdivision.

a) Given that $n-2$ is divisible by 3, construct a graph to show that $f(n) \geq 3n-5$.

b) Prove that $f(n) = 3n-5$ when $n-2$ is divisible by 3 and that otherwise $f(n) = 3n-6$. (Hint: Use induction on n, invoking Exercise 6.2.8 for the 3-connected case.) (Thomassen [1984])

(Comment: Mader [1998] proved the more difficult result that $3n-6$ is the maximum number of edges in an n-vertex simple graph with no K_5-subdivision.)

6.2.11. (!) Let H be a graph with maximum degree at most 3. Prove that a graph G contains a subdivision of H if and only if G contains a subgraph contractible to H.

6.2.12. (!) Wagner [1937] proved that the following condition is necessary and sufficient for a graph G to be planar: neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and contractions of edges.

a) Show that deletion and contraction of edges preserve planarity. Conclude from this that Wagner's condition is necessary.

b) Use Kuratowski's Theorem to prove that Wagner's condition is sufficient.

6.2.13. Prove that a graph G is planar if and only if for every cycle C in G, the conflict graph for C is bipartite. (Tutte [1958])

6.2.14. Let x and y be vertices of a planar graph G. Prove that G has a planar embedding with x and y on the same face unless $G-x-y$ has a cycle C with x and y in conflicting C-fragments in G. (Hint: Use Kuratowski's Theorem. Comment: Tutte proved this without Kuratowski's Theorem and used it to prove Kuratowski's Theorem.)

6.2.15. Let G be a 3-connected simple plane graph containing a cycle C. Prove that C is the boundary of a face in G if and only if G has exactly one C-fragment. (Comment: Tutte [1963] proved this to obtain Whitney's [1933b] result that 3-connected planar graphs have essentially only one planar embedding. See also Kelmans [1981a])

6.2.16. (+) Let G be an outerplanar graph with n vertices, and let P be a set of n points in the plane, no three of which lie on a line. The *extreme points* of P induce a convex polygon that contains the other points in its interior.

a) Let p_1, p_2 be consecutive extreme points of P. Prove that there is a point $p \in P - \{p_1, p_2\}$ such that 1) no point of P is inside $p_1 p_2 p$, and 2) some line l through p separates p_1 from p_2, meets P only at p, and has exactly $i-2$ points of P on the side of l containing p_2.

b) Prove that G has a straight-line embedding with its vertices mapped onto P. (Hint: Use part (a) to prove the stronger statement that if v_1, v_2 are two consecutive vertices of the unbounded face of a maximal outerplanar graph G, and p_1, p_2 are consecutive vertices of the convex hull of P, then G can be straight-line embedded on P such that $f(v_1) = p_1$ and $f(v_2) = p_2$.) (Gritzmann–Mohar–Pach–Pollack [1989])

6.3. Parameters of Planarity

Every property and parameter we have studied for general graphs can be studied for planar graphs. The problem of greatest historical interest is the maximum chromatic number of planar graphs. We will also study parameters that measure how far a graph is from being a planar graph.

COLORING OF PLANAR GRAPHS

Because every simple n-vertex planar graph has at most $3n-6$ edges, such a graph has a vertex of degree at most 5. This yields an inductive proof that planar graphs are 6-colorable (see Exercise 2). Heawood improved the bound.

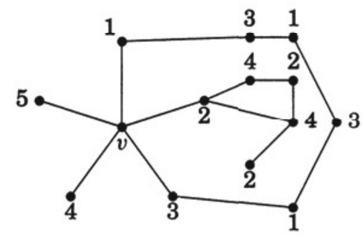
6.3.1. Theorem. (Five Color Theorem—Heawood [1890]) Every planar graph is 5-colorable.

Proof: We use induction on $n(G)$.

Basis step: $n(G) \leq 5$. All such graphs are 5-colorable.

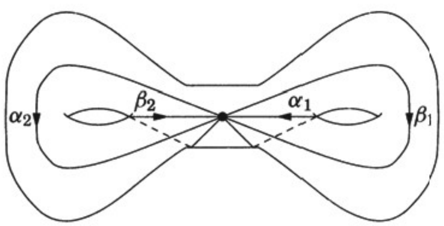
Induction step: $n(G) > 5$. The edge bound (Theorem 6.1.23) implies that G has a vertex v of degree at most 5. By the induction hypothesis, $G-v$ is 5-colorable. Let $f\colon V(G-v) \to [5]$ be a proper 5-coloring of $G-v$. If G is not 5-colorable, then f assigns each color to some neighbor of v, and hence $d(v) = 5$. Let v_1, v_2, v_3, v_4, v_5 be the neighbors of v in clockwise order around v. Name the colors so that $f(v_i) = i$.

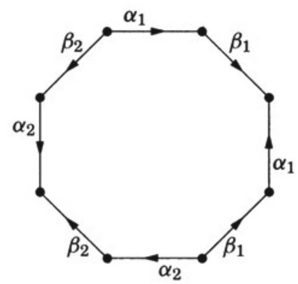
Let $G_{i,j}$ denote the subgraph of $G-v$ induced by the vertices of colors i and j. Switching the two colors on any component of $G_{i,j}$ yields another proper 5-coloring of $G-v$. If the component of $G_{i,j}$ containing v_i does not contain v_j, then we can switch the colors on it to remove color i from $N(v)$. Now giving color i to v produces a proper 5-coloring of G. Thus G is 5-colorable unless, for each choice of i and j, the component of $G_{i,j}$ containing v_i also contains v_j. Let $P_{i,j}$ be a path in $G_{i,j}$ from v_i to v_j, illustrated below for $(i, j) = (1, 3)$.



Consider the cycle C completed with $P_{1,3}$ by v; this separates v_2 from v_4.

a bouquet of 2γ loops. For example, the double torus can also be represented by an octagon with boundary $\alpha\beta\gamma\delta\alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1}$. ∎





6.3.23. Remark. *Euler's Formula for S_γ.* A **2-cell** is a region such that every closed curve in the interior can be continuously contracted to a point. A **2-cell embedding** is an embedding where every region is a 2-cell. Euler's Formula generalizes for 2-cell embeddings of connected graphs on S_γ (Exercise 35) as

$$n - e + f = 2 - 2\gamma.$$

For example, our embedding of K_7 on the torus ($\gamma = 1$) has 7 vertices, 21 edges, 14 faces, and $7 - 21 + 14 = 0$. The proof of Euler's Formula for S_γ is like the proof in the plane, except that the basis case of 1-vertex graphs needs more care. It requires showing that it takes 2γ cuts to lay the surface flat (that is, to obtain a 2-cell embedding of a graph with one vertex and one face). ∎

6.3.24. Lemma. Every simple n-vertex graph embedded on S_γ has at most $3(n - 2 + 2\gamma)$ edges.

Proof: Exercise 35. ∎

Note that K_7 satisfies Lemma 6.3.24 with equality on the torus ($\gamma = 1$), as every face in the toroidal embedding of K_7 is a 3-gon. Hence K_7 is a maximal toroidal graph. Rewriting $e \le 3(n-2+2\gamma)$ yields a lower bound on the number of handles we must add to obtain a surface on which G is embeddable; thus $\gamma(G) \ge 1 + (e - 3n)/6$.

Lemma 6.3.24 leads to an analogue of the Four Color Theorem for S_γ.

6.3.25. Theorem. (Heawood's Formula—Heawood [1890]) If G is embeddable on S_γ with $\gamma > 0$, then $\chi(G) \le \left\lfloor (7 + \sqrt{1 + 48\gamma})/2 \right\rfloor$.

Proof: Let $c = (7 + \sqrt{1 + 48\gamma})/2$. It suffices to prove that every simple graph embeddable on S_γ has a vertex of degree at most $c - 1$; the bound on $\chi(G)$ then follows by induction on $n(G)$. Since $\chi(G) \le c$ for all graphs with at most c vertices, so need only consider $n(G) > c$.

We use Lemma 6.3.24 to show that the average (and hence minimum) degree is at most $c - 1$. The second inequality below follows from $\gamma > 0$ and $n > c$.

Since c satisfies $c^2 - 7c + (12 - 12\gamma) = 0$, we have $c - 1 = 6 - (12 - 12\gamma)/c$, so the average degree satisfies the desired bound.

$$\frac{2e}{n} \le \frac{6(n - 2 + 2\gamma)}{n} \le 6 - \frac{12 - 12\gamma}{c} = c - 1.$$ ∎

The key inequality here fails when $\gamma = 0$. Thus the argument is invalid for planar graphs, even though the formula reduces to $\chi(G) \le 4$ when $\gamma = 0$. Proving that the Heawood bound is sharp involves embedding K_n on S_γ with $\gamma = \lceil (n-3)(n-4)/12 \rceil$. The proof breaks into cases by the congruence class of n modulo 12 (K_7 is the first example in the easy class). Completed in Ringel–Youngs [1968], it comprises the book *Map Color Theorem* (Ringel [1974]).

Having considered the coloring problem on S_γ, one naturally wonders which graphs embed on S_γ. Planar graphs have many characterizations, beginning with Kuratowski's Theorem (Theorem 6.2.2) and Wagner's Theorem (Exercise 6.2.12). On any surface, embeddability is preserved by deleting or contracting an edge. Thus every surface has a list of "minor-minimal" obstructions to embeddability. Wagner's Theorem states that the list for the plane is $\{K_{3,3}, K_5\}$; every nonplanar graph has one of these as a minor.

More than 800 minimal forbidden minors are known for the torus. For each surface, the list is finite; this follows from the much more general statement below (the *subdivision* relation in Kuratowski's Theorem leads to infinite lists).

6.3.26. Theorem. (The Graph Minor Theorem—Robertson–Seymour [1985]) In any infinite list of graphs, some graph is a minor of another. ∎

This is perhaps the most difficult theorem known in graph theory. The complete proof takes well over 500 pages (without computer assistance) in a series of 20 papers stretching beyond the year 2000. It has many ramifications about structure of graphs and complexity of computation. The techniques involved in the proof have spawned new areas of graph theory. Some aspects of these techniques and their relation to the proof of the Graph Minor Theorem are presented in the final chapter of the text by Diestel [1997].

EXERCISES

6.3.1. (–) State a polynomial-time algorithm that takes an arbitrary planar graph as input and produces a proper 5-coloring of the graph.

6.3.2. (–) A graph G is k-**degenerate** if every subgraph of G has a vertex of degree at most k. Prove that every k-degenerate graph is $k + 1$-colorable.

6.3.3. (–) Use the Four Color Theorem to prove that every outerplanar graph is 3-colorable.

6.3.4. (–) Determine the crossing numbers of $K_{2,2,2,2}$, $K_{4,4}$, and the Petersen graph.

• • • • •

6.3.5. Prove that every planar graph decomposes into two bipartite graphs. (Hedetniemi [1969], Mabry [1995])

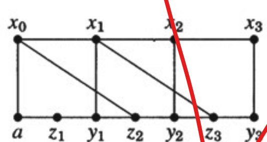
6.3.6. Without using the Four Color Theorem, prove that every planar graph with at most 12 vertices is 4-colorable. Use this to prove that every planar graph with at most 32 edges is 4-colorable.

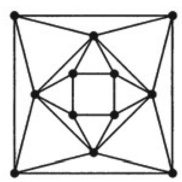
6.3.7. (!) Let H be a configuration in a planar triangulation (Definition 6.3.2). Let H' be obtained by labeling the neighbors of the ring vertices with their degrees and then deleting the ring vertices. Prove that H can be retrieved from H'.

6.3.8. Create a configuration with ring size 5 in a planar triangulation such that every internal vertex has degree at least five.

6.3.9. (+) Prove that every planar configuration having ring size at most four is reducible. (Hint: The ring is a separating cycle C. Prove that if smaller triangulations are 4-colorable, then the C-lobes of G have 4-colorings that agree on C.) (Birkhoff [1913])

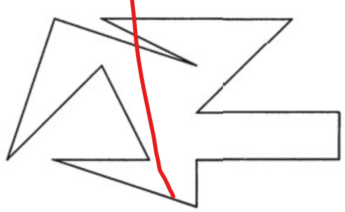
6.3.10. Grötzsch's Theorem [1959] (see Steinberg [1993], Thomassen [1994a]) states that a triangle-free planar graph G is 3-colorable. Hence $\alpha(G) \geq n(G)/3$. Tovey–Steinberg [1993] proved that $\alpha(G) > n(G)/3$ always. Prove that this is best possible by considering the family of graphs G_k defined as follows: G_1 is the 5-cycle, with vertices a, x_0, x_1, y_1, z_1 in order. For $k > 1$, G_k is obtained from G_{k-1} by adding the three vertices x_k, y_k, z_k and the five edges $x_{k-1}x_k, x_ky_k, y_kz_k, z_ky_{k-1}, z_kx_{k-2}$. The graph G_3 is shown on the left below. (Fraughnaugh [1985])





6.3.11. Define a sequence of plane graphs as follows. Let G_1 be C_4. For $n > 1$, obtain G_n from G_{n-1} by adding a new 4-cycle surrounding G_{n-1}, making each vertex of the new cycle also adjacent to two consecutive vertices of the previous outside face. The graph G_3 is shown on the right above. Prove that if n is even, then every proper 4-coloring of G_n uses each color on exactly n vertices. (Albertson)

6.3.12. (!) Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior is visible to some guard. Construct a polygon that requires $\lfloor n/3 \rfloor$ guards. (Chvátal [1975], Fisk [1978])



6.3.13. An *art gallery with walls* is a polygon plus some nonintersecting chords called "walls" that join vertices. Each interior wall has a tiny opening called a "doorway". A guard in a doorway can see everything in the two neighboring rooms, but a guard not in a doorway cannot see past a wall. Determine the minimum number t such that for every walled art gallery with n vertices, it is possible to place t guards so that every interior point is visible to some guard. (Hutchinson [1995], Kündgen [1999])

6.3.14. (+) Prove that a maximal planar graph is 3-colorable if and only if it is Eulerian. (Hint: For sufficiency, use induction on $n(G)$. Choose an appropriate pair or triple of adjacent vertices to replace with appropriate edges.) (Heawood [1898])

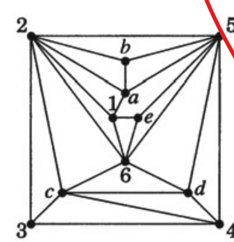
6.3.15. (!) Prove that the vertices of an outerplanar graph can be partitioned into two sets so that the subgraph induced by each set is a disjoint union of paths. (Hint: Define the partition using the parity of the distance from a fixed vertex.) (Akiyama–Era–Gervacio [1989], Goddard [1991])

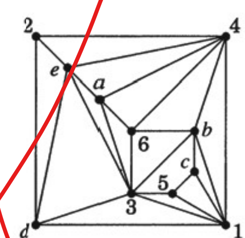
6.3.16. (−) Prove that the 4-dimensional cube Q_4 is nonplanar. Decompose it into two isomorphic planar graphs; hence Q_4 has thickness 2.

6.3.17. Prove that K_n has thickness at least $\lfloor \frac{n+7}{6} \rfloor$. (Hint: $\lceil \frac{x}{r} \rceil = \lfloor \frac{x+r-1}{r} \rfloor$.) Show that equality holds for K_8 by finding a self-complementary planar graph with 8 vertices. (Comment: The thickness equals $\lfloor \frac{n+7}{6} \rfloor$ except that K_9 and K_{10} have thickness 3; Beineke–Harary [1965] for $n \not\equiv 4 \bmod 6$, and Alekseev–Gončakov [1976] for $n \equiv 4 \bmod 6$.)

6.3.18. Decompose K_9 into three pairwise-isomorphic planar graphs.

6.3.19. Prove that if G has thickness 2, then $\chi(G) \leq 12$. Use the two graphs below to show that $\chi(G)$ may be as large as 9 when G has thickness 2. (Sulanke)





6.3.20. (!) When r is even and s is greater than $(r-2)^2/2$, prove that the thickness of $K_{r,s}$ is $r/2$. (Beineke–Harary–Moon [1964])

6.3.21. Determine $\nu(K_{1,2,2,2})$ and use it to compute $\nu(K_{2,2,2,2})$.

6.3.22. Prove that $K_{3,2,2}$ has no planar subgraph with 15 edges. Use this to give another proof that $\nu(K_{3,2,2}) \geq 2$.

6.3.23. Let M_n be the graph obtained from the cycle C_n by adding chords between vertices that are opposite (if n is even) or nearly opposite (if n is odd). The graph M_n is 3-regular if n is even, 4-regular if n is odd. Determine $\nu(M_n)$. (Guy–Harary [1967])

6.3.24. The graph P_n^k has vertex set $[n]$ and edge set $\{ij : |i-j| \leq k\}$. Prove that P_n^3 is a maximal planar graph. Use a planar embedding of P_n^3 to prove that $\nu(P_n^4) = n - 4$. (Harary–Kainen [1993])

6.3.25. For every positive integer k, construct a graph that embeds on the torus but requires at least k crossings when drawn in the plane. (Hint: A single easily described toroidal family suffices; use Proposition 6.3.13.)

6.3.26. (!) Use Kleitman's computation that $\nu(K_{6,n}) = 6\lfloor \frac{n-6}{2} \rfloor \lfloor \frac{n-7}{2} \rfloor$ to give counting arguments for the following lower bounds.

a) $\nu(K_{m,n}) \ge m\frac{m-1}{5}\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. (Guy [1970])

b) $\nu(K_p) \ge \frac{1}{80}p^4 + O(p^3)$.

6.3.27. (!) It is conjectured that $\nu(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. Suppose that this conjecture holds for $K_{m,n}$ and that m is odd. Prove that the conjecture then holds also for $K_{m+1,n}$. (Kleitman [1970])

6.3.28. (!) Suppose that m and n are odd. Prove that in all drawings of $K_{m,n}$, the parity of the number of pairs of edges that cross is the same. (We consider only drawings where edges cross at most once and edges sharing an endpoint do not cross.) Conclude that $\nu(K_{m,m})$ is odd when $m-3$ and $n-3$ are divisible by 3 and even otherwise.

6.3.29. Suppose that n is odd. Prove that in all drawings of K_n, the parity of the number of pairs of edges that cross is the same. Conclude that $\nu(K_n)$ is even when n is congruent to 1 or 3 modulo 8 and is odd when n is congruent to 5 or 7 modulo 8.

6.3.30. (!) It is known that $\nu(C_m \square C_n) = (m-2)n$ if $m \le \min\{5, n\}$. Also $\nu(K_4 \square C_n) = 3n$.

a) Find drawings in the plane to establish the upper bounds.

b) Prove that $\nu(C_3 \square C_3) \ge 2$. (Hint: Find three subdivisions of $K_{3,3}$ that together use each edge exactly twice.)

6.3.31. Let $f(n) = \nu(K_{n,n,n})$.

a) Show that $3\nu(K_{n,n}) \le f(n) \le 3\binom{n}{2}^2$.

b) Show that $\nu(K_{3,2,2}) = 2$ and $\nu(K_{3,3,1}) = 3$. Show that $5 \le \nu(K_{3,3,2}) \le 7$ and $9 \le \nu(K_{3,3,3}) \le 15$.

c) Exercise 6.3.26a shows that the lower bound in part (a) is at least $(3/20)n^4 + O(n^3)$. Improve it by using a recurrence to show that $f(n) \ge n^3(n-1)/6$.

d) The upper bound in part (a) is $\frac{3}{4}n^4 + O(n^3)$. Improve it to $f(n) \le \frac{9}{16}n^4 + O(n^3)$. (Hint: One construction embeds the graph on a tetrahedron and generalizes to a construction for $K_{l,m,n}$; another uses K_n and generalizes to a construction for $K_{n,\dots,n}$.)

6.3.32. (∗) Construct an embedding of a 3-regular nonbipartite simple graph on the torus so that every face has even length.

6.3.33. (∗) Suppose that n is at least 9 and is not a prime or twice a prime. Construct a 6-regular toroidal graph with n vertices.

6.3.34. (∗) An embedding of a graph on a surface is **regular** if its faces all have the same length. Construct regular embeddings of $K_{4,4}$, $K_{3,6}$, and $K_{3,3}$ on the torus.

6.3.35. (∗) Prove Euler's Formula for genus γ: For every 2-cell embedding of a graph on the surface S_γ, the numbers of vertices, edges, and faces satisfy $n - e + f = 2 - 2\gamma$. Conclude that an n-vertex graph embeddable on S_γ has at most $3(n-2+2\gamma)$ edges.

6.3.36. (∗) Use Euler's Formula for S_γ to prove that $\gamma(K_{3,3,n}) \ge n-2$, and determine the value exactly for $n \le 3$.

6.3.37. (∗) For every positive integer k, use Euler's Formula for higher surfaces to prove that there exists a planar graph G such that $\gamma(G \square K_2) \ge k$.

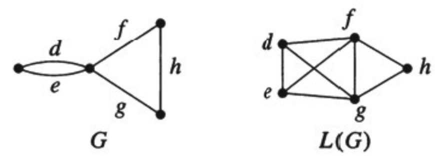
Chapter 7

Edges and Cycles

7.1. Line Graphs and Edge-coloring

Many questions about vertices have natural analogues for edges. Independent sets have no adjacent vertices; matchings have no "adjacent" edges. Vertex colorings partition vertices into independent sets; we can instead partition edges into matchings. These pairs of problems are related via line graphs (Definition 4.2.18). Here we repeat the definition, emphasizing our return to the context in which a graph may have multiple edges. We use "line graph" and $L(G)$ instead of "edge graph" because $E(G)$ already denotes the edge set.

7.1.1. Definition. The **line graph** of G, written $L(G)$, is the simple graph whose vertices are the edges of G, with $ef \in E(L(G))$ when e and f have a common endpoint in G.



Some questions about edges in a graph G can be phrased as questions about vertices in $L(G)$. When extended to all simple graphs, the vertex question may be more difficult. If we can solve it, then we can answer the original question about edges in G by applying the vertex result to $L(G)$.

In Chapter 1, we studied Eulerian circuits. An Eulerian circuit in G yields a spanning cycle in the line graph $L(G)$. (Exercise 7.2.10 shows that the converse need not hold!) In Section 7.2, we study spanning cycles for graphs in general. As discussed in Appendix B, this problem is computationally difficult.

In Chapter 3, we studied matchings. A matching in G becomes an independent set in $L(G)$. Thus $\alpha'(G) = \alpha(L(G))$, and the study of α' for graphs is